

Traveling and standing waves in coupled pendula and Newton's cradle

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Abstract

The existence of traveling and standing waves is investigated for chains of coupled pendula with periodic boundary conditions. The results are proven by applying topological methods to subspaces of symmetric solutions. The main advantage of this approach comes from the fact that only properties of the linearized forces are required. This allows to cover a wide range of models such as Newton's cradle, the Fermi-Pasta-Ulam lattice and the Toda lattice.

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A chain of coupled pendula is a device of hanging limbs from an elastic rod. When this device is set in motion, each limb behaves like a pendulum that interacts with its neighbors by torsion forces. When the forces are approximated by Hooke's law, the equations describing the pendula are equivalent to the discretized Sine-Gordon equations.

Another device of interest is the Newton's Cradle. This system consists of beads suspended by inelastic strings. In the absence of contact between beads, they have pendular motion; when beads collide with their neighbors, they repel each other with a Hertz's type force.

The paper aim is to study a model that includes the mentioned pendula among other situations of interest, such as the Klein-Gordon, Fermi-Pasta-Ulam and Toda lattices.

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The movement of n coupled oscillators, $q_j(t) \in \mathbb{R}$ for $j = 1, \dots, n$, with periodic boundary conditions, $q_j = q_{j+n}$, is described by equations

$$-\ddot{q}_j = U'(q_j) + W'(q_j - q_{j-1}) - W'(q_{j+1} - q_j), \quad (1)$$

where potentials U and W represent the dynamic and nonlinear interaction of the oscillators, respectively.

The dynamic of a pendulum is governed by

$$U(x) = \omega^2(1 - \cos x), \quad \omega^2 = cg/l, \quad (2)$$

where g is the acceleration due to gravity and l is the length of the pendulum. The constant c represents the coupling strength after a rescaling, i.e. the normalized Hooke's Law is given by

$$W(x) = \frac{1}{2}x^2,$$

and the Hertz's contact force by

$$W(x) = \frac{2}{5}|x|^{5/2} \text{ if } x \leq 0, \quad W(x) = 0 \text{ if } x > 0. \quad (3)$$

We assume that equations (1) have homogenous equilibria of the form

$$q_j(t) = a \quad \text{for } j = 1, \dots, n.$$

These properties hold true in the coupled pendula and the Newton's cradle when $a = 0$ and $a = \pi$. Under these considerations, the nonlinear equations (1) have periodic orbits arising from the homogenous equilibria.

The study of periodic orbits in Hamiltonian systems goes back to Poincare. Lyapunov proves the nonlinear continuation of periodic orbits from normal modes of elliptic equilibria under non-resonant conditions. Later on, the Lyapunov center theorem was extended to consider the multiplicity of periodic orbits and the global properties of the families. Regarding multiplicity, the Weinstein–Moser theorem (1973) proves multiple periodic solutions of Hamiltonian systems with fixed energy, while the Fadell–Rabinowitz theorem (1978) considers fixed period. In [1], Alexandre-Yorke (1978) prove the global property of Lyapunov families. The proofs of these theorems make use of topological invariants that consider the S^1 -symmetry induced by time translations. These theorems have been generalized to consider spatio-temporal

symmetries, for instance, see [3, 18] for fixed energy, [3, 6] for fixed period and [2, 7, 15] for global bifurcation. Other equivariant approaches can be found in [5, 12, 23] and references therein.

In the context of equation (1), in [11], local bifurcation of periodic solutions is proven in the case $U = 0$ and $W''(0) \neq 0$, and in [13], a Weinstein-Moser theorem is proven in the case $U'' = 1$ and $W''(0) \neq 0$. The present paper proves the bifurcation of periodic solutions using Brouwer degree in spaces of spatio-temporal symmetric functions, including a \mathbb{Z}_2 -symmetry induced by the reflection in time. In contrast with the results in [11] and [13], this procedure has the advantage that allows to prove the global property.

Theorem 8. Assume $W''(0) \neq 0$, which is the case in the coupled pendula. For each $k \in [1, n/2) \cap \mathbb{N}$ such that

$$\nu_k = \sqrt{U''(a) + (2 \sin k\pi/n)^2 W''(0)} > 0 \quad (4)$$

is non-resonant (Definition 7), the homogenous equilibrium has three global branches of $2\pi/\nu$ -periodic solutions; for $k \in \{n/2, n\}$, only one branch exists. The frequencies ν along the branches converge to ν_k as the solutions approach the equilibrium and the bifurcating branch is a continuum that either goes to infinity in Sobolev norm or period, or ends at other bifurcation point.

In the coupled pendula, at the equilibrium $a = 0$, the non-resonant condition of ν_k holds true except for a finite of parameters ω (Section 4.1). For the resonant parameters the theorem proves only the existence of the branches with the higher frequency $\nu_j = l\nu_k$.

When $W''(0) = 0$ all frequencies are resonant,

$$\nu_k := \sqrt{U''(a)} \quad \text{for } k \in \{1, \dots, n\}.$$

These resonances make impossible to obtain multiple periodic solutions by means of topological degree; instead, the existence of multiple standing waves is proven with the Fadell-Rabinowitz theorem for odd potentials given in [6].

Theorem 11. Assume $W''(0) = 0$ and $U''(a) > 0$, which is the case in Newton's cradle. The homogenous equilibrium has at least $n/2 - 1$ bifurcations of $2\pi/\nu$ -periodic solutions with symmetries (6) and (7). The frequency ν of the periodic solutions is arbitrarily close to ν_0 , but the bifurcation does not necessarily form a local continuum.

The periodic solutions have the symmetries of traveling and standing waves. This fact is proven exploiting the equivariance of equations (1) under the action of the group

$$D_n \times O(2),$$

where D_n is composed by permutations of the oscillators and $O(2)$ shifts and reflects time; see Definition 1.

The symmetries presented in Section 3 are valid along the global branches, while the estimates shown in Section 2.1 are valid locally. We reproduce here the simple case n odd and $k = 1$. The symmetries for $k = 1$ have been analyzed previously in [11] and [12]. The present paper completes the classification of the symmetries for all k 's.

Symmetries and estimates. Let

$$\zeta = 2\pi/n, \quad q_j(t) = a + x_j(\nu t),$$

where $x_j(t)$ is 2π -periodic and ν is the frequency. The branch of traveling waves has symmetries and local estimates,

$$\begin{aligned} x_j(t) &= x_{n-j}(-t) = x_{j+1}(t - \zeta), \\ x_j(t) &= r \cos(t + j\zeta) + \mathcal{O}(r^2), \end{aligned} \tag{5}$$

where r is a parameterization of the branch and $\mathcal{O}(r^2)$ is a 2π -periodic function. The branch of standing waves has symmetries and local estimates,

$$\begin{aligned} x_j(t) &= x_{n-j}(t) = x_j(-t), \\ x_j(t) &= r \cos(j\zeta) \cos t + \mathcal{O}(r^2), \end{aligned} \tag{6}$$

and the other,

$$\begin{aligned} x_j(t + \pi) &= x_{n-j}(t) = x_j(-t), \\ x_j(t) &= r \sin(j\zeta) \sin t + \mathcal{O}(r^2). \end{aligned} \tag{7}$$

Traveling waves for Newton's cradle have been estimated asymptotically in [16], and for beads in [17] and [22]. In these papers, traveling waves are constructed by means of a reduction to a single equation with delay. This procedure is commonly used in many problems; see [19] and the references therein. However, the reduction to one equation cannot be used to prove existence of standing waves, and then, one of the achievements of the present paper is the construction of them for the Newton's cradle. Neither the Weinstein–Moser theorem in [13] is applicable to the Newton's cradle because it assumes non-resonant conditions over $W''(0)$. Through Proposition 14, a Weinstein–Moser theorem can be proven for the Newton's cradle.

In the case of beads, $W'''(0) = 0$ and $U = 0$, the topological approach used to establish the existence of standing waves cannot be used due to the

fact that $\nu_k = 0$ for $k = 1, \dots, n$. In [17] is shown that standing waves exist even for homogenous potential W with $\nu_k = 0$. Therefore, further work is necessary to investigate the existence of standing waves in the beads problem.

In Section 1, we set the bifurcation problem and make a global reduction to a finite number of Fourier components. In Section 2.1, we prove the global bifurcation in the case $W''(0) \neq 0$. In Section 2.2, we prove existence of standing waves in the case $W''(0) = 0$ and $U''(a) > 0$. In Section 3, we describe the symmetries. In Section 4, we apply the theorems to the coupled pendula and the FPU and Toda lattices. In Section 4.3, we present a comment about the existence of standing waves in the case of a homogenous potential W with $\nu_k = 0$.

1 Setting up the problem

Let $q = (q_1, \dots, q_n)$ and

$$V(q) = \sum_{j=1}^n [U(q_j) + W(q_j - q_{j-1})]. \quad (8)$$

Equations (1) can be expressed in vectorial form as

$$-\ddot{q} = \nabla V(q).$$

Hereafter, we assume that the potential V is twice differentiable and

$$\mathbf{a} = (a, \dots, a)$$

is an equilibrium, $\nabla V(\mathbf{a}) = 0$. This is equivalent to assume that $U'(a) = 0$ and $W'(0) = 0$.

Using the change of variables $q(t) = \mathbf{a} + x(\nu t)$, the system of equations become

$$-\nu^2 \ddot{x} = \nabla V(\mathbf{a} + x).$$

Let $H_{2\pi}^2(\mathbb{R}^n)$ be the Sobolev space of 2π -periodic functions. We define the operator f from $H_{2\pi}^2(\mathbb{R}^n)$ in $L_{2\pi}^2(\mathbb{R}^n)$ as

$$f(x; \nu) = -\nu^2 \ddot{x} - \nabla V(\mathbf{a} + x). \quad (9)$$

Since \mathbf{a} is an equilibrium, then $f(0; \nu) = 0$ for all ν . Therefore, the branches of $2\pi/\nu$ -periodic solutions emanating from the equilibrium \mathbf{a} correspond to zeros of $f(x; \nu)$ bifurcating from $(0, \nu_0)$.

Definition 1 Let D_n be the subgroup of permutations generated by

$$\zeta(j) = j + 1, \quad \kappa(j) = n - j \quad \text{modulus } n.$$

Let $\rho : D_n \times O(2) \rightarrow GL(L_{2\pi}^2)$ be the homomorphism generated by

$$\rho(\gamma)(x_1, \dots, x_n) = (x_{\gamma(1)}, \dots, x_{\gamma(n)}), \quad (10)$$

for $\gamma \in D_n$ and for $\varphi, \bar{\kappa} \in O(2)$,

$$\rho(\varphi)x(t) = x(t + \varphi), \quad \rho(\bar{\kappa})x(t) = x(-t). \quad (11)$$

Then ρ defines a $D_n \times O(2)$ -representation of $L_{2\pi}^2(\mathbb{R}^n)$ and induces the left action in $L_{2\pi}^2(\mathbb{R}^n)$ given by $\rho(\gamma, x) = \rho(\gamma)x$.

Since $V(x)$ is invariant by the action of D_n , then $\nabla V(x)$ is D_n -equivariant. Thus, the operator $f(x)$ is D_n -equivariant. Given that the equations are autonomous and reversible in time, then $f(x)$ is $D_n \times O(2)$ -equivariant.

1.1 Lyapunov-Schmidt reduction

In the case $W''(0) \neq 0$, we will prove existence of periodic solutions using a global Lyapunov-Schmidt reduction and Brouwer degree. Although an application of Leray–Schauder degree can provide similar results without reductions, we prefer this approach because the reduction is required in the Newton’s cradle anyway. The idea of the global Lyapunov-Schmidt reduction is taken from [15].

The Fourier expansion of $x \in L_{2\pi}^2$ and the projection P are defined as

$$x(t) = \sum_{l \in \mathbb{Z}} x_l e^{ilt} \text{ and } Px = \sum_{|l| \leq l_0} x_l e^{ilt}.$$

Let \mathbf{x}_1 and \mathbf{x}_2 be the components of x , given by

$$\mathbf{x}_1 = Px, \quad \mathbf{x}_2 = (I - P)x.$$

The components of f are given by $f_1 = Pf$ and $f_2 = (I - P)f$.

We realize the global Lyapunov-Schmidt reduction in the set $\Omega_\rho \times \Lambda_\varepsilon$, where

$$\Omega_\rho = \{x \in H_{2\pi}^2 : \|x\|_{H_{2\pi}^2} < \rho\}, \quad \Lambda_\varepsilon = \{\nu > \varepsilon\}.$$

Proposition 2 *There is a l_0 such that $\mathbf{x}_2(\mathbf{x}_1, \nu)$ is the only solution of $f_2(\mathbf{x}_1 + \mathbf{x}_2, \nu) = 0$ in $\Omega_\rho \times \Lambda_\varepsilon$. Thus $f(\mathbf{x}_1 + \mathbf{x}_2, \nu) = 0$ if and only if $\phi(\mathbf{x}_1, \nu) = 0$, where*

$$\begin{aligned}\phi(\mathbf{x}_1, \nu) &= f_1(\mathbf{x}_1 + \mathbf{x}_2(\mathbf{x}_1, \nu); \nu) \\ &= -\nu^2 \partial_{tt} \mathbf{x}_1 - P \nabla V(\mathbf{a} + \mathbf{x}_1 + \mathbf{x}_2(\mathbf{x}_1, \nu)).\end{aligned}\tag{12}$$

Furthermore, the reduced map $\phi(\mathbf{x}_1, \nu)$ is $D_n \times O(2)$ -equivariant, where the action of $\varphi, \bar{\kappa} \in O(2)$ in the l -th Fourier component is given by

$$\rho(\varphi)x_l = e^{il\varphi}x_l, \quad \rho(\bar{\kappa})x_l = \bar{x}_l.$$

Proof. If we find a positive constant α such that

$$\|\partial_{\mathbf{x}_2} f_2(\mathbf{x}_1 + \mathbf{x}_2)y_2\|_{L_{2\pi}^2} \geq \alpha \|y_2\|_{H_{2\pi}^2}$$

for all $(x, \nu) \in \Omega_\rho \times \Lambda_\varepsilon$, the global implicit function theorem due to Hadamard (Theorem 5.1.5 in [4]) implies existence of a unique function $\mathbf{x}_2(\mathbf{x}_1, \nu)$ such that $f_2(\mathbf{x}_1 + \mathbf{x}_2(\nu, \mathbf{x}_1), \nu) = 0$. Using the uniqueness of $\mathbf{x}_2(\mathbf{x}_1, \nu)$, it can be proven that $f_1(\mathbf{x}_1 + \mathbf{x}_2(\mathbf{x}_1, \nu), \nu)$ is $D_n \times O(2)$ -equivariant.

Let $y_2 \in (I - P)H_{2\pi}^2$. Since $V \in C^2(\mathbb{R}^n)$ and $\|x\|_{C_{2\pi}^0} < c\rho$ for all $x \in \Omega_\rho$, then $\|D^2V(\mathbf{a} + x)\|_{C_{2\pi}^0} < C\rho$ for all $x \in \Omega_\rho$. Therefore,

$$\|(I - P)D^2V(\mathbf{a} + x)y_2\|_{L_{2\pi}^2} \leq C\rho \|y_2\|_{L_{2\pi}^2} \leq C(\rho/l_0^2) \|y_2\|_{H_{2\pi}^2}.$$

Using the previous estimate and $\|\nu^2 \partial_{tt} y_2\|_{L_{2\pi}^2} \geq \varepsilon^2 \|y_2\|_{H_{2\pi}^2}$ for $\nu \in \Lambda_\varepsilon$, we have

$$\|\partial_{\mathbf{x}_2} f_2 y_2\|_{L_{2\pi}^2} \geq \|\nu^2 \partial_{tt} y_2\|_{L_{2\pi}^2} - \|(I - P)D^2V(\mathbf{a} + x)y_2\|_{L_{2\pi}^2} \geq \alpha \|y_2\|_{H_{2\pi}^2},$$

where $\alpha = \varepsilon^2 - C(\rho/l_0^2)$. We conclude that $\alpha > 0$ if $l_0 > C\sqrt{\rho}/\varepsilon$. Note that the number of Fourier components l_0 goes to infinity as $\varepsilon \rightarrow 0$ and $\rho \rightarrow \infty$.

■

Using Taylor's expansion of the map $f_2(\mathbf{x}_1 + \mathbf{x}_2)$, we obtain the estimate

$$\|\mathbf{x}_2(\mathbf{x}_1, \nu)\|_{H_{2\pi}^2} \leq c \|\mathbf{x}_1\|^2\tag{13}$$

for x close to 0. Thus, the linearization of the reduced map at $(0, \nu)$ is

$$\phi'(0; \nu)\mathbf{x}_1 = -\nu^2 \partial_{tt} \mathbf{x}_1 - D^2V(\mathbf{a})\mathbf{x}_1 = \sum_{|l| \leq l_0} M(l\nu)x_l,\tag{14}$$

where

$$M(l\nu) = (l\nu)^2 I - D^2V(\mathbf{a}).$$

1.2 Irreducible representations

In this section we identify the irreducible representations of $O(2) \times D_n$. In the l -th Fourier component, the action of $O(2)$ is

$$\rho(\varphi)x_l = e^{li\varphi}x_l, \quad \rho(\bar{\kappa})x_l = \bar{x}_l.$$

The Fourier components are subrepresentations of the group $O(2)$; then, we need to find the irreducible representations of $x_l \in \mathbb{C}^n$ under the action of D_n .

Let $\zeta = 2\pi/n$ and $e_k \in \mathbb{C}^n$ be

$$e_k = n^{-1/2}(e^{1(ik\zeta)}, e^{2(ik\zeta)}, \dots, e^{n(ik\zeta)}). \quad (15)$$

The vectors e_k for $k = 1, \dots, n$ are orthonormal and their direct sum is the whole space \mathbb{C}^n . Therefore, we can expand $x_l = \sum_{k \in \mathbb{Z}_n} x_{k,l} e_k$, where $\mathbb{Z}_n = \{1, \dots, n\}$, and

$$x(t) = \sum_{(k,l) \in \mathbb{Z}_n \times \mathbb{Z}} x_{k,l} e_k e^{ilt}.$$

Proposition 3 *For $k = n/2, n$ the action of the group $D_n \times O(2)$ in $x_{k,1} \in \mathbb{C}$ is given by*

$$\rho(\zeta, \varphi)x_{k,1} = \pm e^{i\varphi}x_{k,1}, \quad \rho(\kappa)x_{k,1} = x_{k,1}, \quad \rho(\bar{\kappa})x_{k,1} = \bar{x}_{k,1}, \quad (16)$$

with negative sign for $k = n/2$. For $k \in [1, n/2) \cap \mathbb{N}$, the action in $(x_{k,1}, x_{n-k,1}) \in \mathbb{C}^2$ is given by

$$\begin{aligned} \rho(\zeta, \varphi)(x_{k,1}, x_{n-k,1}) &= e^{i\varphi}(e^{ik\zeta}x_{k,1}, e^{-ik\zeta}x_{n-k,1}), \\ \rho(\kappa)(x_{k,1}, x_{n-k,1}) &= (x_{n-k,1}, x_{k,1}), \\ \rho(\bar{\kappa})(x_{k,1}, x_{n-k,1}) &= (\bar{x}_{n-k,1}, \bar{x}_{k,1}). \end{aligned} \quad (17)$$

Proof. The actions of ζ and κ in e_k are

$$\begin{aligned} \rho(\zeta)e_k &= n^{-1/2}(e^{2(ik\zeta)}, e^{3(ik\zeta)}, \dots, e^{n(ik\zeta)}, e^{(ik\zeta)}) = e^{ik\zeta}e_k, \\ \rho(\kappa)e_k &= (e^{(n-1)ik\zeta}, e^{(n-2)ik\zeta}, \dots, e^{2ik\zeta}, e^{1ik\zeta}, e^{nik\zeta}) = e_{n-k}. \end{aligned}$$

Moreover, the action of $\bar{\kappa} \in O(2)$ is

$$\rho(\bar{\kappa})e_k z = n^{-1/2}(e^{-ik\zeta}\bar{z}, e^{-2ik\zeta}\bar{z}, \dots, e^{-(n-1)ik\zeta}\bar{z}, e^{-nik\zeta}\bar{z}) = e_{n-k}\bar{z}.$$

The result follows. ■

By the previous proposition, the subspaces generated by e_n , $e_{n/2}$ and $e_k \oplus e_{n-k}$ for $k \in [1, n/2) \cap \mathbb{N}$ are irreducible representations.

1.3 Isotropy groups

In this section, we identify the maximal isotropy groups of the irreducible representations presented in Proposition 3.

The action of $D_n \times O(2)$ in the subspaces generated by e_n and $e_{n/2}$ are given by (16), where the minus sign corresponds to the representation generated by $e_{n/2}$. The subspace $x_{n,1} \in \mathbb{R}$ has isotropy group

$$T_n = \langle (\zeta, 0), (\kappa, 0), (0, \bar{\kappa}) \rangle, \quad (18)$$

and the subspace $x_{n/2,1} \in \mathbb{R}$,

$$T_{n/2} = \langle (\zeta, \pi), (\kappa, 0), (0, \bar{\kappa}) \rangle. \quad (19)$$

Both isotropy groups T_k for $k = n, n/2$ have fixed point spaces of dimension one.

Definition 4 *Let h be the maximum common divisor of k and n ,*

$$\bar{k} = \frac{k}{h} \text{ and } \bar{n} = \frac{n}{h}.$$

The cases $k \in [1, n/2) \cap \mathbb{N}$ are analyzed in the appendix. We reproduce here the main results.

Proposition 5 *For $k \in [1, n/2) \cap \mathbb{N}$, the representation $e_k \oplus e_{n-k}$ has three maximal isotropy groups with fixed point spaces of dimension one. An isotropy group is*

$$T_k = \langle (\zeta, -k\zeta), (\kappa, \bar{\kappa}), (\bar{n}\zeta, 0) \rangle.$$

For \bar{n} odd, the other isotropy groups are

$$S_k = \langle (\kappa, 0), (0, \bar{\kappa}), (\bar{n}\zeta, 0) \rangle, \quad \tilde{S}_k = \langle (\kappa, \pi), (0, \pi\bar{\kappa}), (\bar{n}\zeta, 0) \rangle.$$

The case \bar{n} even is given in the appendix.

These isotropy groups are relevant because the reduced map

$$\phi^H(x; \nu) : \text{Fix}(H) \rightarrow \text{Fix}(H)$$

is well defined and under non-resonant conditions, the linearization $D\phi^H(0, \nu)$ has a simple eigenvalue in the kernel.

1.4 Linearization

The Hessian of V is

$$D^2V(\mathbf{a}) = U''(a)I + W''(0)A,$$

where $A = (a_{i,j})_{i,j=1}^n$ is the matrix defined by $a_{i,j} = 2$ if $i = j$, $a_{i,j} = -1$ if $|i - j| = 1$ modulus n , and $a_{i,j} = 0$ otherwise.

Proposition 6 *The matrix $M(\nu) = \nu^2 I - D^2V(\mathbf{a})$ is diagonal in the basis $\{e_k\}$,*

$$x = \sum_{k \in \mathbb{Z}_n} x_k e_k, \quad M(\nu)x = \sum_{k \in \mathbb{Z}_n} \lambda_k(\nu) x_k e_k, \quad (20)$$

where the eigenvalues are

$$\lambda_k(\nu) = \nu^2 - U''(a) - (2 \sin k\zeta/2)^2 W''(0). \quad (21)$$

Proof. Since the j -th coordinate of e_k is $n^{-1/2} e^{ij\zeta}$, then the j -th coordinate of Ae_k is

$$(2 - (e^{-ik\zeta} + e^{ik\zeta}))n^{-1/2} e^{ij\zeta} = 4 \sin^2(k\zeta/2) n^{-1/2} e^{ij\zeta}.$$

We conclude that $Ae_k = 4 \sin^2(k\zeta/2) e_k$ and

$$M(\nu)e_k = [\nu^2 I - U''(a)]e_k - W''(0)Ae_k = \lambda_k(\nu)e_k.$$

■

The fact that $\lambda_{n-k} = \lambda_k$ for $k \in [1, n/2) \cap \mathbb{N}$ is consequence of Schur's lemma. That is, since $e_k \oplus e_{n-k}$ is an irreducible representation, Schur's lemma implies that $M(\nu) = \lambda I$ in $e_k \oplus e_{n-k}$.

In the basis $x_{k,l}$, we have

$$\phi'(0; \nu) \mathbf{x}_1 = \sum_{|l| \leq l_0} \sum_{k \in \mathbb{Z}_n} \lambda_k(l\nu) x_{k,l} e_k e^{ilt}. \quad (22)$$

2 Main results: Bifurcation theorems

We proceed to prove two bifurcation theorems.

2.1 Coupled pendula

If $W''(0) \neq 0$, we avoid resonant frequencies by assuming that $D^2V(\mathbf{a})$ is invertible, which holds if $U''(a)$ and $W''(0)$ are positive.

Definition 7 *We say that the frequency*

$$\nu_k = \sqrt{U''(a) + (2 \sin k\pi/n)^2 W''(0)} \quad (23)$$

is non-resonant if $l\nu_k \neq \nu_j$ for $j \in (k, n/2] \cap \mathbb{N}$ and $l \geq 2$.

Theorem 8 *Assume $W''(0) \neq 0$ and $D^2V(\mathbf{a})$ is invertible. For each $k \in [0, n/2] \cap \mathbb{N}$ such that $\nu_k > 0$ is non-resonant, the equilibrium \mathbf{a} has three global bifurcations of $2\pi/\nu$ -periodic solutions emanating from $\nu = \nu_k$ with isotropy groups T_k , S_k , and \tilde{S}_k .*

Proof. The linear map $\phi'(0; \nu)$ has l -th Fourier block $M(l\nu)$ with eigenvalues

$$\lambda_j(l\nu) = (l\nu)^2 - \nu_j^2,$$

for $j \in [0, n/2] \cap \mathbb{N}$.

For $l = 0$, the matrix $M(0) = -D^2V(\mathbf{a})$ is invertible by hypothesis. For $l \geq 2$, since $W''(0) \neq 0$, frequencies ν_j are increasing in j . Thus $l\nu_k > \nu_j$ for $j \in [0, k] \cap \mathbb{N}$ and, by hypothesis, $l\nu_k \neq \nu_j$ for $j \in (k, n/2] \cap \mathbb{N}$. Therefore, matrices $M(l\nu_k)$ are invertible for $l \geq 2$. Given that $M(l\nu)$ is continuous in ν , the blocks $M(l\nu)$ are invertible for ν close to ν_k .

For $l = 1$, since ν_k is increasing in $k \in [0, n/2] \cap \mathbb{N}$, then $\nu_k^2 - \nu_j^2 \neq 0$ for $j \neq k$. Therefore, the linearization $\phi'(0; \nu)$ is invertible for ν close to ν_k , except for the block that corresponds to the representation $(x_{k,1}, x_{n-k,1}) \in e_k \oplus e_{n-k}$,

$$\lambda_k(\nu)I : \mathbb{C}^2 \rightarrow \mathbb{C}^2. \quad (24)$$

Set H equal to T_k , S_k or \tilde{S}_k . Since the group H has fixed point spaces of real dimension equal to one in $e_k \oplus e_{n-k}$, the restriction $D\phi^H(0; \nu_k)$ has a simple eigenvalue crossing zero in (24). Using Brouwer degree as in Theorem 14 in [9] or [15], we conclude existence of a local bifurcation in the fixed point space of H .

The global property follows from assuming that the branch is contained in the set $\Omega_\rho^H \times \Lambda_\varepsilon$ for some ε and ρ , unless it is an unbounded continuum set with period or Sobolev norm going to infinite. Applying Brouwer degree

to the reduction in $\Omega_p^H \times \Lambda_\varepsilon$, we conclude that the sum of the local degrees at the bifurcation points is zero, as in Theorem 5.2 in [15] or Theorem 15 in [9]. ■

Remark 9 *In the case $U = 0$, the Hessian $D^2V(\mathbf{a})$ has a zero-eigenvalue corresponding to the conserved quantity $\sum_{j=1}^n q_j = 0$. If this is the only zero-eigenvalue, we can extend the previous theorem using the restriction of f to the subspace*

$$X = \{x \in L_{2\pi}^2 : \sum_{j=1}^n x_j = 0\}.$$

Also, analogous theorems can be proven in the case of D_n -equivariant long range interactions, $W(q_1, \dots, q_n)$.

Under non-resonant considerations, the local reduction can be realized on the 1-th Fourier component

$$\mathbf{x}_1 = \sum_{|l|=1} x_l e^{ilt}.$$

Let r be a parameterization of the amplitude of the local branch, where the frequency ν is a function of r . The branch with isotropy group T_k has eigenvalues corresponding to the coordinates $(x_{k,1}, x_{n-k,1}) = (r, 0)$ (see the appendix). We conclude that the projected component \mathbf{x}_1 can be estimated by

$$\mathbf{x}_1(t) = r(e^{it}e_k + \overline{e^{it}e_k}) + \mathcal{O}(r^2).$$

Moreover, since $e^{it}e_k + \overline{e^{it}e_k} = 2\cos(t + jk\zeta)$ and $\mathbf{x}_2(\mathbf{x}_1; \nu) = \mathcal{O}(r^2)$, then

$$x_j(t) = 2r\cos(t + jk\zeta) + \mathcal{O}(r^2), \quad (25)$$

where $\mathcal{O}(r^2)$ is a 2π -periodic function of order r^2 .

Similarly, the coordinates for the isotropy group S_k are $(x_{k,1}, x_{n-k,1}) = (r, r)$, then

$$\mathbf{x}_1(t) = r(e^{it}e_k + \overline{e^{it}e_k} + e^{it}e_{n-k} + \overline{e^{it}e_{n-k}}) + \mathcal{O}(r^2).$$

We conclude

$$x_j(t) = 4r\cos jk\zeta \cos t + \mathcal{O}(r^2). \quad (26)$$

For the group \tilde{S}_k and n odd, the coordinates are $(x_{k,1}, x_{n-k,1}) = (r, -r)$, then

$$x_j(t) = -4r \sin jk\zeta \sin t + \mathcal{O}(r^2). \quad (27)$$

For n even, the coordinates are $(x_{k,1}, x_{n-k,1}) = (r, re^{i\zeta})$, then

$$x_j(t) = -4r \sin(jk\zeta - \zeta/2) \sin(t + \zeta/2) + \mathcal{O}(r^2). \quad (28)$$

Note that for n odd, since $\sin(2\pi j/n) = 0$ for $j \in \{n, n/2\} \cap \mathbb{N}$ and $\cos(2\pi j/n) \neq 0$, standing waves S_k and \tilde{S}_k have nodes of different orders, r and r^2 respectively.

2.2 Newtons's cradle

For Newtons's cradle, W is given by (3) and $U(x) = \omega^2(1 - \cos x)$. Since $\nu_k = \omega$ for $k = 1, \dots, n$, the previous theorem cannot provide the existence of many periodic solutions. In this case, the application of Weinstein–Moser theorem guaranties the existence of at least n periodic solutions in each constant energy surface,

$$H(q, p) = \frac{1}{2} \|p\|^2 + V(q). \quad (29)$$

However, these solutions may agree with the n traveling waves found in [16], and the symmetries have to be considered in order to obtain new solutions.

Definition 10 *Let $S = S^1$ and \tilde{S} be the group generated by (κ, π) ($0, \pi\bar{\kappa}$) if n is odd, and $(\kappa\zeta, 0)$ and $(0, \zeta\bar{\kappa})$ if n is even.*

We will prove existence of standing waves using the fixed point spaces of the groups S and \tilde{S} , which are the isotropy groups S_1 and \tilde{S}_1 without the generator (π, π) .

Since $\nu_k = \omega$ for all k , the 1-Fourier component is non-resonant with other Fourier components. Thus the local reduction of Section 1.1 can be realized on the 1-th Fourier component,

$$\mathbf{x}_1 = x_1 e^{it} + \bar{x}_1 e^{-it},$$

and the local bifurcation map is defined by $\phi(\mathbf{x}_1; \nu) : \mathbb{C}^n \rightarrow \mathbb{C}^n$ for (x, ν) close to $(0, \omega)$.

Since the operator $f(x)$ is the gradient of

$$F(x) = \int_0^{2\pi} \left(\frac{\nu^2}{2} |\partial_t x|^2 - V(x) \right) dx,$$

we can conclude (see Section 1.9 in [15]) that $\phi(\mathbf{x}_1)$ is the gradient of the reduced potential

$$\Phi(\mathbf{x}_1) = F(\mathbf{x}_1 + \mathbf{x}_2(\mathbf{x}_1)).$$

That is, $\Phi'(\mathbf{x}_1)h = \langle \phi(\mathbf{x}_1), h \rangle_{L^2_{2\pi}}$.

Since Lyapunov-Schmidt reductions preserve equivariant properties, the potential $\Phi(x_1, \nu)$ is $D_n \times O(2)$ -invariant. Setting H equal to S or \tilde{S} , this implies that the restriction $\Phi^H : \text{Fix}(H) \rightarrow \mathbb{R}$ is well defined and the gradient is

$$\nabla \Phi^H = \phi^H : \text{Fix}(H) \rightarrow \text{Fix}(H).$$

Furthermore, the element $\pi \in O(2)$ is contained in the Weyl group of H and acts multiplying by -1 the fixed point space of H (see appendix), then $\Phi^H(x_1) = \Phi^H(-x_1)$.

Using $\mathbf{a} = 0$, $V(0) = 0$, $D^2V(0) = \omega^2 I$ and $\mathbf{x}_2(\mathbf{x}_1; \nu) = \mathcal{O}(|\mathbf{x}_1|^2)$, we estimate

$$\Phi(x_1, \nu) = 2\pi(\nu^2 - \omega^2) |x_1|^2 + o(|x_1|^2). \quad (30)$$

Therefore, the Fadell-Rabinowitz theorem in [6] implies that the odd potential $\Phi^H(\mathbf{x}_1, \nu)$ has at least $\dim \text{Fix}(H)$ branches of critical points. Since $\dim \text{Fix}(H) \geq n/2 - 1$ for H equal to S and \tilde{S} (see the appendix), the following theorem holds.

Theorem 11 *Set H equal to S or \tilde{S} . If $(\mathbf{x}_1, \nu) = (0, \omega)$ is an isolated point of the potential $\Phi^H(\mathbf{x}_1, \nu)$, then $f(x, \nu)$ has at least d_- zeros for $\nu < \omega$ and d_+ for $\nu > \omega$ in the fixed point space of H . The zeros converge to $(0, \omega)$ as $\nu \rightarrow \omega$ and*

$$d_- + d_+ \geq n/2 - 1.$$

Therefore, there are at least $n/2 - 1$ bifurcations of periodic solutions with symmetries S and \tilde{S} emanating from the homogeneous equilibria. The information about the symmetries of these $n/2 - 1$ bifurcations can be improved if one considers similar procedures in the fixed point spaces of S_k and \tilde{S}_k .

Another proof of the previous theorem can be given with \mathbb{Z}_2 -equivariant Conley index. The idea in [3] consists on using the invariant property of the Conley index and the change in the unstable set of the gradient flow generated by Φ^H , which has dimension $\dim \text{Fix}(H)$ for $\nu < \omega$ and 0 for $\nu > \omega$.

Using Proposition 14 and Theorem 9.9 in [3], we can prove that the energy surface $H^{-1}(\varepsilon)$ for small ε contains at least $n/2 - 1$ solutions with isotropy groups S and \tilde{S} (Weinstein–Moser theorem). These kinds of solutions are called brake orbits in [3]. A Weinstein–Moser theorem for coupled pendula was proven in [13] using results of [18].

3 Description of symmetries

The isotropy groups T_k , S_k and \tilde{S}_k have the generator $(\bar{n}\zeta, 0)$. Functions fixed by $(\bar{n}\zeta, 0)$ satisfy

$$x_j(t) = x_{j+\bar{n}}(t),$$

where h is the maximum common divisor of k and n , $\bar{k} = k/h$ and $\bar{n} = n/h$.

Therefore, the oscillators in these solutions form a wave of length \bar{n} that is repeated h times along the pendula. In the following discussion, we describe only the wave of length \bar{n} ,

$$(x_1, \dots, x_{\bar{n}}).$$

3.1 Traveling waves

The group T_k has generators $(\zeta, -k\zeta)$ and $(\kappa, \bar{\kappa})$, solutions with isotropy group T_k have symmetries

$$x_j(t) = x_{n-j}(-t) = x_{j+1}(t - \bar{k}(2\pi/\bar{n})), \quad (31)$$

For $\bar{k} = 1$, two consecutive oscillators have a phase shift of $2\pi/\bar{n}$,

$$x_j(t) = x_{j+1}(t - 2\pi/\bar{n}).$$

For $\bar{k} \neq 1$, the solutions are just permutations of the case $\bar{k} = 1$.

3.2 Standing waves of the first kind

Standing waves have different behavior depending on the parity of \bar{n} and $\bar{n}/2$. We present three cases for each kind of standing waves. Given that cases $k \neq 1$ are permutations of $\bar{k} = 1$, we present only the case $\bar{k} = 1$.

Hereafter, we denote

$$x_j^*(t) = x_j(t + \pi).$$

\bar{n} odd

Since the isotropy group S_k has generators $(\kappa, 0)$ and $(0, \kappa)$, functions fixed by S_k have symmetries

$$x_j(t) = x_{\bar{n}-j}(t) = x_j(-t).$$

Setting $\bar{n} = 2m + 1$, the wave of length \bar{n} is

$$(x_0, x_1, \dots, x_m, x_m, \dots, x_1), \quad (32)$$

where x_j are even functions for $j = 0, \dots, m$.

$\bar{n}/2$ odd

The isotropy group S_k has generators $(\kappa, 0)$, $(\frac{\bar{n}}{2}\zeta, \pi)$ and $(0, \bar{\kappa})$, solutions in the fixed point space of S_k satisfy

$$x_j(t) = x_{\bar{n}-j}(t) = x_{j+\bar{n}/2}^*(t) = x_j(-t).$$

Setting $\bar{n} = 4m + 2$, the wave of length \bar{n} is

$$(x_0, x_1, \dots, x_m, x_m^*, \dots, x_1^*, x_0^*, x_1^*, \dots, x_m^*, x_m, \dots, x_1), \quad (33)$$

where x_j are even functions for $j = 0, \dots, m$.

$\bar{n}/2$ even

Setting $\bar{n} = 4m + 4$, we conclude that the wave of length \bar{n} is

$$(x_0, x_1, \dots, x_m = x_m^*, \dots, x_1^*, x_0^*, x_1^*, \dots, x_m^* = x_m, \dots, x_1), \quad (34)$$

where x_j are even functions for $j = 0, \dots, m$ and x_m is π -periodic.

Observe that, although solutions (33) and (34) have the same isotropy group, the solutions are qualitatively different at the oscillator x_m .

3.3 Standing waves of the second kind

\bar{n} odd

The isotropy group \tilde{S}_k has generators (κ, π) and $(0, \pi\bar{\kappa})$. Functions fixed by \tilde{S}_k have symmetries

$$x_j(t) = x_{\bar{n}-j}(t + \pi) = x_j(\pi - t).$$

Setting $\bar{n} = 2m + 1$, the wave of length \bar{n} is

$$(x_0, x_1, \dots, x_m, x_m^*, \dots, x_1^*), \quad (35)$$

where $x_j(\cdot + \pi/2)$ are even functions for $j = 1, \dots, m$ and $x_0(t)$ is even π -periodic.

$\bar{n}/2$ odd

Since the isotropy group \tilde{S}_k has generators $(\kappa\zeta, 0)$, $(\frac{\bar{n}}{2}\zeta, \pi)$ and $(0, \zeta\bar{\kappa})$, solutions in the fixed point space of \tilde{S}_k satisfy

$$x_j(t) = x_{\bar{n}-(j+1)}(t) = x_{\bar{n}/2+j}^*(t) = x_j(2\pi/n - t).$$

Setting $\bar{n} = 4m + 2$, the wave of length \bar{n} is

$$(x_0, x_1, \dots, x_m = x_m^*, \dots, x_1^*, x_0^*, x_0^*, x_1^*, \dots, x_m^* = x_m, \dots, x_1, x_0), \quad (36)$$

where $x_j(\pi/n + \cdot)$ are even for $j = 0, \dots, m$ and x_m is π -periodic.

$\bar{n}/2$ even

For $\bar{n} = 4m + 4$, we conclude that solutions with isotropy group \tilde{S}_k satisfy

$$(x_0, \dots, x_m, x_m^*, \dots, x_0^*, x_0^*, \dots, x_m^*, x_m, \dots, x_0), \quad (37)$$

where $x_j(\pi/n + \cdot)$ are even for $j = 0, \dots, m$.

Solutions (36) and (37) are different close to the oscillator x_m .

4 Applications

In most applications the potential W is convex. If U is also convex, the frequencies ν_k are always positive; if U is concave, the frequencies ν_k are positive for $k \in [k_0, n/2] \cap \mathbb{N}$.

4.1 Coupled pendula

The coupled pendula via torsion springs are governed by the dynamics of $U(x) = \omega^2(1 - \cos x)$ and $W(x) = x^2/2$. Since $U'(0) = \omega^2 \sin x$ and $W''(0) = 1$, the homogenous equilibria are $0 = (0, \dots, 0)$ and $\pi = (\pi, \dots, \pi)$.

Given that $U''(0) = \omega^2$ and $U''(\pi) = -\omega^2$, the condition that gives the existence of bifurcation is

$$\nu_k = \sqrt{\pm\omega^2 + (2 \sin k\pi/n)^2} > 0,$$

where the minus sign correspond to π . For the 0 equilibrium, the frequencies ν_k are always positive. For the π equilibrium, despite the fact that the uncoupled system is unstable, the coupled system has $2\pi/\nu_k$ -periodic solutions near the equilibrium for $k \in [k_0, n/2] \cap \mathbb{N}$, where

$$k_0 \sim (n/\pi) \arcsin(\omega/2).$$

In Theorem 8, the non-resonant condition of ν_k is equivalent to $\omega^2 \neq \pm\omega_l(j)$ for integers $l \geq 1$ and $j > k$, where

$$\omega_l(j) := -\frac{(2 \sin k\pi/n)^2 - (2 \sin j\pi/n)^2/l^2}{1 - 1/l^2}. \quad (38)$$

For the 0 equilibrium, the non-resonant condition $\omega^2 \neq \omega_l(j)$ holds true except for a finite number of resonant parameters $\omega_l(j)$, and for π , except for countable number of parameters $-\omega_l(j) \rightarrow 2 \sin k\pi/n$ as $l \rightarrow \infty$.

The same statements are true for the discrete Klein-Gordon equation with potentials $U = \omega^2 x^2 + x^3$ and $W(x) = x^2/2$ in [20], and for the bistable potential $U(x) = \omega^2(1 - x^2)^2/4$ for equilibria with $a = 0$ and $a = \pm 1$.

4.2 FPU and Toda lattices

In this section, we consider oscillators with $U(x) = 0$ and nonlinear interactions

$$W(x) = x^2/2 + \sum_{k=3}^{\infty} \beta_k x^k.$$

The FPU lattice corresponds to $W(x) = x^2/2 + \beta x^3/3$ and Toda lattice to $W(x) = e^{-x} + x - 1$.

Since $W'(0) = 1$, then 0 is an homogeneous equilibrium. Therefore, for each $k \in [1, n/2] \cap \mathbb{N}$ such that

$$\nu_k = 2 \sin k\pi/n$$

is non-resonant, the equilibrium has three global bifurcating branches of periodic solutions. The non-resonant condition $\nu_j \neq l\nu_k$ is equivalent to $\omega_l(j) \neq 0$.

Actually, in [21] is proven that resonances ($\nu_j = l\nu_k$) and higher order resonances do exist. In these cases, Theorem 8 only proves the existence of the bifurcations with the higher frequency $\nu_j = l\nu_k$. In [14] a two dimensional family of superposed traveling waves is constructed for resonances ($\nu_j = l\nu_k$) but in the context of infinite FPU lattices.

4.3 Homogeneous lattices

In this section we discuss the existence of standing waves when $\nu_k = 0$ for all k 's. We reproduce the case $U = 0$ and $W(x) = \frac{2}{5}|x|^{5/2}$ presented in [17].

Equations (1) have standing waves of the form $q_j(t) = a_j q(t)$ when

$$-\ddot{q} = W'(q) \tag{39}$$

and

$$-a_j = W'(a_{j+1} - a_j) - W'(a_j - a_{j-1}). \tag{40}$$

Since $W(x)$ is convex, equation (39) has only periodic solutions.

Let $b_j = W'(a_j - a_{j-1})$ be the momentum of a_j . Equation (40) is equivalent to

$$-a_j = b_{j+1} - b_j, \quad a_j - a_{j-1} = b_j |b_j|^{-1/3}.$$

Let $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be

$$\phi(a, b) = \left(a + b |b|^{-1/3}, b - (a + b |b|^{-1/3}) \right). \tag{41}$$

The orbits $(a_j)_{j \in \mathbb{Z}}$ of ϕ are solutions of (40).

In [16] is shown that the map ϕ is conservative and has many invariant orbits near $(0, 0)$. These invariant orbits correspond to periodic and quasiperiodic (in space) standing waves. These arguments exhibit existence of standing waves even in the case that topological methods cannot be applied. Therefore, further investigations are required to establish existence of standing waves for the beads problem studied in [16] and [22].

Appendix

The action of $D_n \times O(2)$ in the irreducible representation $(z_1, z_2) \in e_1 \oplus e_{n-1}$ is given by

$$\begin{aligned}\rho(\zeta, \varphi)(z_1, z_2) &= e^{i\varphi}(e^{i\zeta}z_1, e^{-i\zeta}z_2), \\ \rho(\kappa)(z_1, z_2) &= (z_2, z_1), \\ \rho(\bar{\kappa})(z_1, z_2) &= (\bar{z}_2, \bar{z}_1).\end{aligned}\tag{42}$$

Proposition 12 *The representation (42) has three maximal isotropy groups with fixed point spaces of real dimension equal to one. The isotropy groups are:*

Parity	Orbit points	Generators	Isotropy group
$n = \text{any}$	$(r, 0)$	$(\zeta, -\zeta), (\kappa, \bar{\kappa})$	T_1
$n = \text{odd}$	(r, r)	$(\kappa, 0), (0, \bar{\kappa})$	S_1
	$(r, -r)$	$(\kappa, \pi), (0, \pi\bar{\kappa})$	\tilde{S}_1
$n = \text{even}$	(r, r)	$(\kappa, 0), (0, \bar{\kappa}), (\pi, \pi)$	S_1
	$(r, re^{i\zeta})$	$(\kappa\zeta, 0), (0, \zeta\bar{\kappa}), (\pi, \pi)$	\tilde{S}_1

Proof. We need to analyze isotropy groups of orbit types. By applying κ , we can assume that $z_1 \neq 0$, unless both coordinates are zero. Using the action of S^1 , we can assume that $(z_1, z_2) = (r, \rho e^{i\psi})$ with $r > 0$. Moreover, by the action

$$\rho(l\zeta, -l\zeta)(r, \rho e^{i\psi}) = (r, \rho e^{i(\psi-2l\zeta)}),$$

we have that $\psi \in (-\zeta, \zeta]$, and by the action of $(\kappa, \bar{\kappa})$ that $\psi \in [0, \zeta]$. Therefore, we need to analyze only isotropy groups of

$$(z_1, z_2) = (r, \rho e^{i\psi}) \text{ for } \psi \in [0, \zeta].$$

Actually, for n odd, the element 2ζ generate the group \mathbb{Z}_n and we can take $\psi \in [0, \zeta/2]$.

The elements $(l\zeta, \varphi), (\kappa l\zeta, \varphi) \in D_n \times S^1$ act as

$$\begin{aligned}\rho(l\zeta, \varphi)(r, \rho e^{i\psi}) &= (re^{i(l\zeta+\varphi)}, \rho e^{i(\psi-l\zeta+\varphi)}), \\ \rho(\kappa l\zeta, \varphi)(r, \rho e^{i\psi}) &= (\rho e^{i(\psi-l\zeta+\varphi)}, re^{i(l\zeta+\varphi)}).\end{aligned}$$

The first coordinate r is fixed by $(l\zeta, \varphi)$ if $\varphi = -l\zeta$, and by $(\kappa l\zeta, \varphi)$ if $\varphi = l\zeta - \psi \bmod 2\pi$. These elements act as

$$\rho(l\zeta, -l\zeta)(r, \rho e^{i\psi}) = (r, \rho e^{i(\psi-2l\zeta)}), \quad (43)$$

$$\rho(\kappa l\zeta, l\zeta - \psi)(r, \rho e^{i\psi}) = (\rho, r e^{i(2l\zeta-\psi)}). \quad (44)$$

Thus, we need to find out when these elements fix the second coordinate.

For $\rho = 0$, elements $(l\zeta, -l\zeta)$ always fix $(r, 0)$. For $\rho \neq 0$, elements $(l\zeta, -l\zeta)$ fix $(r, \rho e^{i\psi})$ if $\psi - 2l\zeta = \psi$ or $l\zeta = \pi \pmod{2\pi}$. That is, the element (π, π) is in the isotropy group of $(r, \rho e^{i\psi})$ when n is even.

For $\rho \neq r$, elements $(\kappa l\zeta, l\zeta - \psi)$ never fix these points. For $\rho = r$, elements $(\kappa l\zeta, l\zeta - \psi)$ fix $(r, r e^{i\psi})$ when $2l\zeta - \psi = \psi \pmod{2\pi}$ or $\psi = l\zeta \pmod{\pi}$. Then $(\kappa\psi, 0)$ is in the isotropy group depending on the parity of n . If n is odd, the element $(\kappa, 0)$ fixes $\psi = 0$, and (κ, π) fixes the point $\psi = \pi$. If n is even, the element $(\kappa, 0)$ fixes $\psi = 0$, and $(\kappa\zeta, 0)$ fixes the point $\psi = \zeta$.

We conclude that the orbit points in the table are fixed by the generators. Moreover, the fixed point spaces of the isotropy groups are the set of orbit points (in the table) for $r \in \mathbb{R}$. Therefore, in the irreducible representation, the fixed point spaces have real dimension equal to one. ■

The action of $D_n \times O(2)$ in the irreducible representation $e_k \oplus e_{n-k}$ is given in Proposition 3. This action is similar to (42), except that $\zeta \in D_n$ acts as

$$\rho(\zeta)(z_1, z_2) = (e^{ik\zeta} z_1, e^{-ik\zeta} z_2).$$

Proposition 13 *For $k \in [1, n/2) \cap \mathbb{N}$, the representation $e_k \oplus e_{n-k}$ has three isotropy groups with fixed point spaces of real dimension equal to one. The isotropy groups are:*

Parity	Orbit P	Generators	Isotropy G
$\bar{n} = \text{any}$	$(r, 0)$	$(\zeta, -k\zeta), (\kappa, \bar{\kappa}), (\bar{n}\zeta, 0)$	T_k
$\bar{n} = \text{odd}$	(r, r)	$(\kappa, 0), (0, \bar{\kappa}), (\bar{n}\zeta, 0)$	S_k
	$(r, -r)$	$(\kappa, \pi), (0, \pi\bar{\kappa}), (\bar{n}\zeta, 0)$	\tilde{S}_k
$\bar{n} = \text{even}$	(r, r)	$(\kappa, 0), (0, \bar{\kappa}), (\frac{\bar{n}}{2}m\zeta, \pi), (\bar{n}\zeta, 0)$	S_k
	$(r, r e^{im\zeta})$	$(\kappa m\zeta, 0), (0, m\zeta\bar{\kappa}), (\frac{\bar{n}}{2}m\zeta, \pi), (\bar{n}\zeta, 0)$	\tilde{S}_k

where $m \in \mathbb{N}$ is the modular inverse of \bar{k} ($m\bar{k} = 1 \bmod \bar{n}$). Furthermore, the Weyl groups of these isotropy groups contain the element $\pi \in S^1$ that acts multiplying by -1 the fixed point spaces.

Proof. The action of $\zeta \in D_n$ can be expressed as

$$\rho(\zeta)(z_1, z_2) = (e^{i\bar{k}(2\pi/\bar{n})} z_1, e^{-i\bar{k}(2\pi/\bar{n})} z_2).$$

The factor \bar{k} acts as a permutation because \bar{k} and \bar{n} are coprime numbers. Since $m\bar{k} = 1$ modulus \bar{n} , then

$$\rho(m\zeta) = (e^{i(2\pi/\bar{n})}, e^{-i(2\pi/\bar{n})}), \rho\left(\frac{\bar{n}}{2}m\zeta\right) = -I, \rho(\bar{n}\zeta) = I.$$

The results in the table can be obtained using the previous proposition with $m\zeta = 2\pi/\bar{n}$ instead of ζ . Furthermore, the element $\pi \in O(2)$ leaves invariant the fixed point spaces and acts as -1 . Then π is contained in the Weyl groups. ■

The 1-th Fourier mode $x_1 \in \mathbb{C}^n$ is the direct sum of the irreducible representations e_1 , $e_{n/2}$ and $e_k \oplus e_{n-k}$ for $k \in [1, n/2) \cap \mathbb{N}$. Let

$$\text{Fix}(H) = \{x_1 \in \mathbb{C}^n : \rho(\gamma)x_1 = x_1, \gamma \in H\}.$$

Proposition 14 *For n odd,*

$$\dim_{\mathbb{R}} \text{Fix}(S) = n/2 + 1/2, \quad \dim_{\mathbb{R}} \text{Fix}(\tilde{S}) = n/2 - 1/2, \quad (45)$$

and for n even,

$$\dim_{\mathbb{R}} \text{Fix}(S) = n/2 + 1, \quad \dim_{\mathbb{R}} \text{Fix}(\tilde{S}) = n/2. \quad (46)$$

Proof. For n odd, the group S is generated by $(\kappa, 0)$ and $(0, \bar{\kappa})$ and \tilde{S} by (κ, π) and $(0, \pi\bar{\kappa})$. For $k \in [1, n/2) \cap \mathbb{N}$, the point $(z_1, z_2) \in e_k \oplus e_{n-k}$ is fixed by S if $z_1 = z_2 \in \mathbb{R}$ and by \tilde{S} if $z_1 = -z_2 \in \mathbb{R}$. Then, the fixed point space of the groups S and \tilde{S} have dimension one in $e_k \oplus e_{n-k}$. We conclude that the total dimension of the fixed point space for the representations $k \in [1, n/2) \cap \mathbb{N}$ is $(n-1)/2$. For $k = n$, the point $x_{n,1} \in \mathbb{C}$ is fixed by S if $x_{n,1} \in \mathbb{R}$, and by \tilde{S} if $x_{n,1} = 0$.

For n even, the group \tilde{S} is generated by $(\kappa\zeta, 0)$ and $(0, \zeta\bar{\kappa})$. For $k \in [1, n/2) \cap \mathbb{N}$, we have the actions

$$\begin{aligned} \rho(0, \zeta\bar{\kappa})(z_1, z_2) &= (e^{i\zeta} \bar{z}_2, e^{i\zeta} \bar{z}_1), \\ \rho(\kappa\zeta, 0)(z_1, z_2) &= (e^{-ik\zeta} z_2, e^{ik\zeta} z_1). \end{aligned}$$

Thus, the point $(z_1, z_2) \in e_k \oplus e_{n-k}$ is fixed by S if $z_1 = z_2 \in \mathbb{R}$, and by \tilde{S} if $z_1 = e^{i\zeta} \bar{z}_2$ and $z_1 = e^{i(k+1)\zeta} z_2$. Both conditions define subspaces of dimension equal to one. Therefore, the total dimension of the fixed point space, for $k \in [1, n/2) \cap \mathbb{N}$, is $n/2 - 1$. For $k = n/2, n$, the point $x_{k,1} \in \mathbb{C}$ is fixed by S if $x_{k,1} \in \mathbb{R}$. Moreover, the representation $k = n$ is fixed by \tilde{S} if $x_{n,1} = re^{i\zeta}$.

■

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